Exercise Session for Financial Data Analysis

Summer term 2011

Problem Set 1: Basic Properties of ARMA Time Series Models

These problems are for presentation in the exercise session on May 31. Write to `haas@stat.uni-muenchen.de`. You can also indicate multiple exercises (ordered according to your preference) in case your most preferred problem has already been assigned.

Problem 1 Three time series were simulated from three different $\text{ARMA}(p, q)$ processes, and their autocorrelation functions (ACFs) and partial autocorrelation functions (PACFs) were estimated by means of their sample analogues. These sample ACFs and PACFs are displayed in Figure 1 for series 1, 2, and 3 in the top, middle, and bottom panel, respectively (see the last page of the sheet). For each series, the left plot shows the ACF, and the right plot shows the PACF. Try to infer the orders of the processes, i.e., p and q, from the ACFs and PACFs. Provide a justification for your answers.

Problem 2[1] For an MA(1) process, $Y_t = \theta \epsilon_{t-1} + \epsilon_t$, show that $|\rho(1)| \leq 1/2$ for any number θ, where $\rho(\tau)$ is the autocorrelation function at lag τ. For which values of θ does $\rho(1)$ attain its maximum and minimum? Sketch $\rho(1)$ as a function of θ.

Problem 3 Consider a stationary AR(2) process,

$$Y_t = \phi_0 + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \epsilon_t, \quad \epsilon_t \overset{iid}{\sim} \text{N}(0, \sigma^2).[2] \tag{1}$$

(i) Derive the Yule–Walker equations and compute $\rho(1)$ and $\rho(2)$, where $\rho(\tau)$ is the autocorrelation function of $\{Y_t\}$ at lag τ, i.e.,

$$\rho(\tau) := \text{Corr}(Y_t, Y_{t-\tau}).$$

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[1]Taken from Shumway and Stoffer, *Time Series Analysis and Its Applications*, Springer, 2006, p. 165.

[2]Here and in the following, $\epsilon_t \overset{iid}{\sim} \text{N}(0, \sigma^2)$ means that the ϵ_ts are generated independently from a normal distribution with mean 0 and variance σ^2.

(ii) Find the variance of Y_t.

(iii) Use the Yule–Walker equations derived in part (i) to suggest a simple estimator of the parameters ϕ_1 and ϕ_2 of model (1) by equating theoretical and sample autocorrelations. (Denote the sample autocorrelations by $\widehat{\rho}(\tau)$). This is an example of the *Yule–Walker estimator* for AR processes.

Problem 4 Consider the AR(2) process

$$Y_t = 0.7Y_{t-1} - 0.1Y_{t-2} + \epsilon_t, \quad \epsilon_t \overset{iid}{\sim} N(0, \sigma^2), \quad t \in \mathbb{Z}.$$

(i) Is this process stationary? If so, calculate the autocorrelation function of Y_t.

Problem 5 Find the variance and the autocorrelation function (ACF) of a stationary ARMA(1,1) process,

$$Y_t = \phi Y_{t-1} + \theta \epsilon_{t-1} + \epsilon_t, \quad \epsilon_t \overset{iid}{\sim} N(0, \sigma^2),$$

and compare the ACF with that of a pure AR(1) process with respect to its flexibility regarding the level and the decay rate of the ACF.

Problem 6 (2 bonus points on top) Consider the AR(1) process

$$Y_t = \phi Y_{t-1} + \epsilon_t, \quad \epsilon_t \overset{iid}{\sim} N(0, \sigma^2), \quad t = 1, 2, \ldots,$$

and assume that it has been initialized in period 0 by the (deterministic) value Y_0.

(i) Find the mean and the variance of Y_t, $t \geq 1$. Is Y_t covariance stationary?

(ii) Show that the autocovariance function is

$$\text{Cov}(Y_t, Y_{t+\tau}) = \sigma^2 \phi^\tau \frac{1 - \phi^{2t}}{1 - \phi^2}, \quad \tau \geq 0. \tag{2}$$

(iii) Argue that the process is "asymptotically stationary" in the sense that, for large t,

$$\begin{aligned}
\text{E}(Y_t) &\approx 0, \\
\text{Var}(Y_t) &\approx \frac{\sigma^2}{1 - \phi^2}, \\
\text{Corr}(Y_t, Y_{t+\tau}) &\approx \phi^\tau.
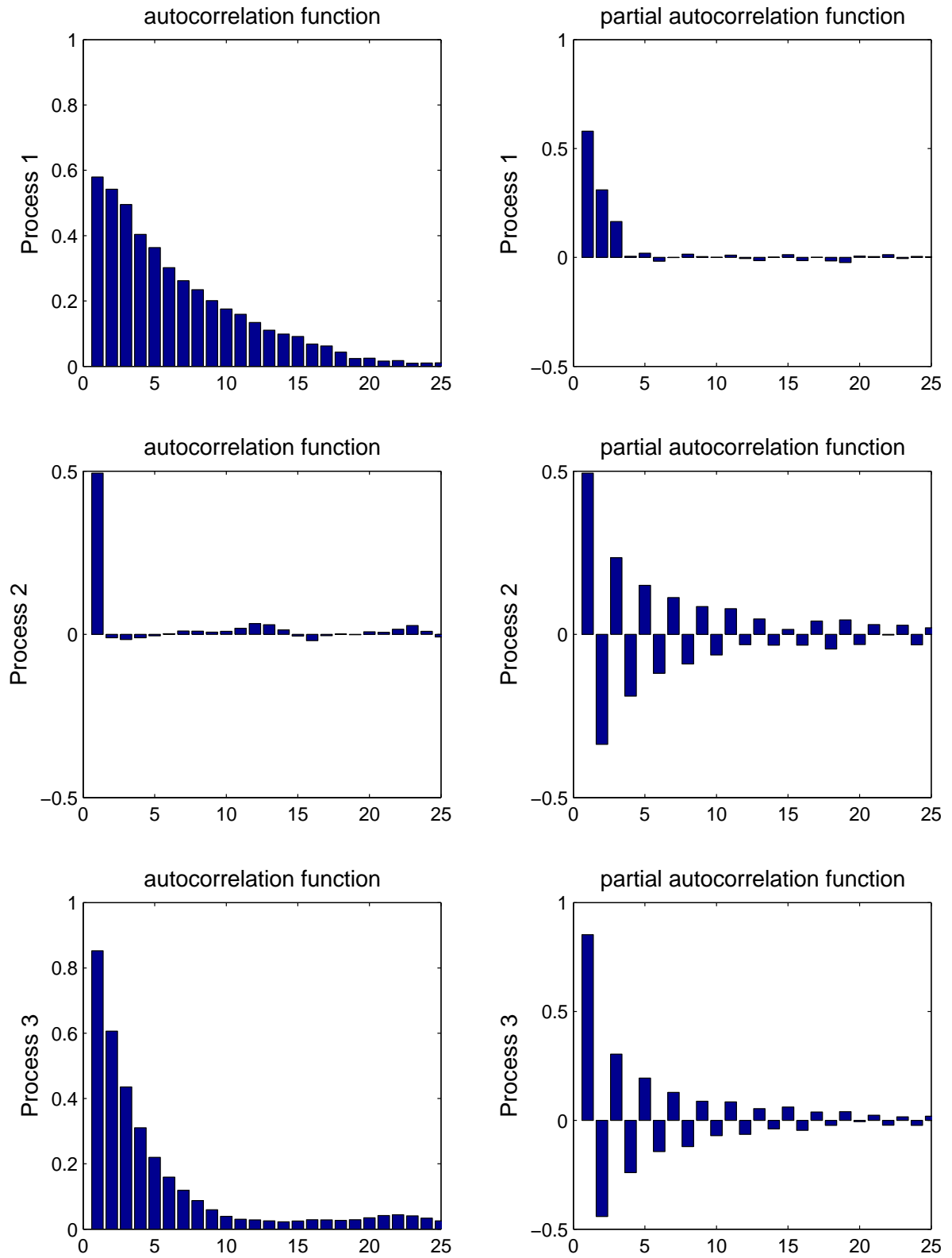\end{aligned}$$

Figure 1: The top, middle, and bottom parts of the figure show the (sample) ACFs (left panel) and PACFs (right panel) of processes 1, 2, and 3, respectively.